ISOLATION OF TORSIONAL VIBRATIONS IN ROTATING MACHINERY

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ABSTRACT

Torsional vibration response of rotating machinery must
be determined when designing an equipment train. Accu-
rate response prediction requires sophisticatsd analysis
techniques which include considertion of all forcing
functions in the system in addition to the mass – elastic
properties of the shafts and couplings. Matrix methods
utilizing the eigenvalue solution provide an adequate
solution technique. A proper understanding of the
assumptions related to the solution technique is essen-
tial. Such areas as lumping, branched system inter-
pretation, gear tooth flexibility, and coupling para-
meters are but a few examples where assumptions must be
made which are essential to a proper solution. Isolation
of torional vibrations can be accomplished once the
system torsional response is properly defined. Case
histories utilizing this technique are presented to
illustrate methods of torsional isolation.

INTRODUCTION

Torsional vibration problems are being encountered in an
ever increasing number due to the expanded use of high
speed rotary equipment. These problems are difficult
to recognize in their primary stages, and many times
they first manifest themselves when a failure occurs.
Typically, this type of failure results in a substantial
penalty in terms of plant downtime, involving first an

analysis of the source of the problem, and then the definition and implementation of a suitable treatment.

To diagnose and minimize the occurrence of possible failures due to torsional vibration, several types of analysis may be employed.

The object of this paper is to present the most practical and accurate method for calculating both the torsional resonant frequency and the forced vibration response. With such analyses, stress levels can be calculated and compared to a failure criterion. In addition, this gaper presents one method which has been successfully used to evaluate the effect of gear stiffness on torsional response. Although gear stiffness is often neglected or improperly simulated in some instances, field experience shows its effect are often critical.

Case histories are included to give example data and to illustrate how a forced vibration analysis can be used to predict equipment reliability or to aid in failure solution.

HOLZER ANALYSIS

The method most commonly employed to calculate the torsional resonant frequency of shafting systems is the Holzer Analysis, chiefly because it is conveniently adaptable to hand calculations. The method requires successive estimates of the shaft resonant frequency as input, and it is possible to overlook solutions if the selected frequency increment is too large. The Holzer tabulation method is useful for a cursory check of most shaft systems; however, it becomes quite cumbersome and tedious when a forced vibration response of complex systems is required. Forced vibration problems can be made with the Holzer Analysis; however, this method does not account fcr the phese of the forcing torques in a camplex forcing function; therefore, an error can exist. This error is slight for relatively ccnstant torque machines; however, in reciprocating compressors and internal combustion engines, the torque is decidedly unsteady dae to piston stroke, and substantial errors can therefore be incurred.

Another limitation of the Holzer Analysis is the inability to properly simulate multiple branced geared systems. The dynamically equivalent system presupposes that the gear assembly is torsionally rigid, which is

approximately the case in large industrial gear boxes
with generous factors of safety. For less conservative-
ly designed systems, gear tooth flexibility has a defi-
nite influence on the torsional frequencies and should
be considered.

EIGENVECTOR ‾ EIGENVALUE MATRIX SOLUTION

A more powerful and versatile method for solving torsional
resonant frequencies is the eigenvector-eigenvalue matrix
solution. While this method insures the calculation of
all the possible modes of vibration, it requires the use
of a digital computer as the computations are numerous
and complex. To calculate the tarsional resonant fre-
quencies of a system, a mathematical model must be syn-
thesized, which will respond in the same manner as the
actual system. All of the elastic, mass and damping
properties of the system are necessary to set up the
mathematical model. Usually, these elastic properties
and the masss inertia can be calculated, measured or ob-
tained from the manufacturer of the element.

Sample differential equations of motion have been written
and are included in Table I. These differential equa-
tions can be converted into a matrix equation for sim-
plicity of solution, and the general form of the equation
would be:

$$[J] \, [\ddot{\theta}] = -[K] \, [\theta]$$

where [J] is the diagonalized mass matrix and [K] is the
stiffness matrix. These matrixes are shown in Table II.
Complex periodic motion may be reduced to individual
harmonics which can be handled easier without compro-
mising the rigorousness of the solution. The equation
for simple harmonic motion can then be assumed for the
general solution.

$$\theta = A \sin \omega t$$

The following relationship for $\ddot{\theta}$ can be obtained by differentiation

$$\theta = A\omega \cos wt$$

$$\dot{\theta} = -A\omega^2 \sin \omega t$$

$$\ddot{\theta} = -\omega^2$$

By substitution, the matrix equation can be rewritten

$$[-J] [\omega^2] [\theta] = -[K] [\theta]$$

where the $[w^2]$ represents the diagonalized eigenvalue matrix which will be called $[\lambda]$. This yields:

$$[-J] [\lambda] [\theta] = -[K] [\theta]$$

By multiplying both sides of the matrix equation by $[J]^{-1}$, the following equation results:

$$[-\lambda] [\theta] = -[J]^{-1} [K] [\theta]$$

and

$$\left\{[J]^{-1} [K] - [\lambda]\right\}[\theta] = 0$$

This form of the matrix is the eigenvalue equation. The values for $[\lambda]$ for which the equation is soluble are known as the characteristic values, or eigenvalues, of the matrix. The vector solutions for $[\theta]$ are the eigenvectors of the matrix $[J]^{-1} [K]$ which shall be referred to as the stiffness-mass matrix.

Physically, the eigenvector represents the mode shape of the vibration, The corresponding eigenvalue represents the vibrational frequency squared. In general, the characteristic equation will have "n" roots with "n" corresponding eigenvectors for a system with "n" rotating masses.

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The solution for the eigenvalues of a problem with five masses will be a fifth order equation whose solution will give five roots of the characteristic equation. Each of these five roots will then represent a resonant frequency squared. The characteristic equation of this example is as follows:

$$A\lambda^5 + B\lambda^4 + C\lambda^3 + D\lambda^2 + E\lambda + F = 0$$

(where A, B. C, D, E, and F represent complex functions of mass and stiffness. The roots of this equation are the eigenvalues from the eigenvalue problem).

The eigenvectors are obtained from the original eigenvalue equation. A set of equations can be derived by using the stiffness—mass matrix, multiplying by an unknown eigen-vector, and forcing this to equal to the product of the eigenvalue and the unknown eigenvector. This set of equations can be solved for eigenvectorc, which represent the vibration mode shape corresponding to the eigenvalue or resonant frequency.

The eigenvector method of solving for the torsional resonant frequency and mode shapes enables the calculation of the forced vibrational response of the system due to various forcing functions at different mass lociititns, including the phasing of all forces. The damping of the system must also be included to insure proper resonant frequencies. An harmonic Fourier expansion of the forc-ing function can be applied at any mass location. In this manner, a complex forcing function with an arbitrary number of harmonics can be simulated.

Systems containing several pinions driven by one gear which cannot be simulated readily by simple hand calcu-lation methods can be solved by the matrix method. The stiffness matrix can be modified by additional off diagonal terms which influence the deflection of the branch point mass. These additional terms are best determined by writing the torsional equations of motion in a syste-matic method. The matrix equations containing the stiff-ness matrix for a branched gear system will be solved in the same manner.

Once the system has been modeled properly and forcing functions applied, the amplitudes at each mass location are known. Since relative deflection between the masses determines the stresses in the shaft, stresses can be calculated for each harmonic, and with proper phasing, the complex peak—to—peak stress wave can be generated which allows for the calculation of the maximum overall peak—to—peak stress.

GEAR TORSIONAL STIFFNESS

Theory

In many geared systems the effect of gear stiffness can
exhibit a critical influence on calculating the torsional
natural frequencies of the system. The effect of gear
stiffness is usually increasingly **important** as the dy-
namic load increases above the normal transmitted static
torque load[f], particularly when the gear is located at a
dynamic torque maximum in the torsional mode shape.

Gear stiffness can only be approximated, since there are
several parameters varying during rotation. These include:

  Instantaneous point of **contact**

  Direction of applied force

  Number of teeth in contact

The following discussion will be limited to spur and
bevel gears, since they represent a large majority of
cases where gear stiffness is critical. The tooth pro-
file is approximated by an isoceles triangle, and Figure
1 illustrates the approximation and notation used. The
linear flexibility is calculated from the equation[2].

$$\frac{1}{K} = \frac{12}{YL}\left(\frac{h}{T}\right)^3\left[2.303\log_{10}\left|\frac{h}{h-h_p}\right| - \frac{h_p}{2h}\left(1+\frac{h_p}{2h}\right)\right] + \frac{h_p}{GLT\left(1-\frac{h_p}{2h}\right)}$$

This value must be calculated for each gearwheel and a
correction factor must be applied to account for the
following second order effects:

(a)   Depression of the tooth surface at the line of contact.

(b)   Flexibility of wheel body adjacent to the tooth.

(c)   Deformation of wheel body.

Experience has shown the following correction factors are applicable:

(a) $R = 1.3$ for plain spur gears.

(b) $R = 1.25$ for bevel gears.

(c) $R = 1.0$ for internal spur gears.

Using these values, the linear flexibility as related to each gear wheel be-

comes:

$$\frac{1}{K_1'} = R\frac{1}{K_1}, \quad \frac{1}{K_2'} = R\frac{1}{K_2}, \quad K_{total} = \frac{1}{K_1'} + \frac{1}{K_2'}$$

The torsional stiffness of the gear system must be related to the torque on the pinion or bullgear. Assuming that the load is shared equally between the two teeth, the following expression can relate the flexibility to the appropriate torque.

$$K_F = 2\ r^2\ K_{total}$$

The torque related stiffness is in reference to the gear wheel with radius r. This stiffness can thereby be applied as a stiffness between the two gear wheel mass inertias. The **following** sample analysis indicates the influence of gear stiffness upon the torsional frequency.

FIELD CASE

In a recent startup at a chemical processing plant there were repeated cooling fan gear box failures. The failures occurred after various lengths of operation ranging from 3-48 hours. A torsiograph was installed on one of the fan systems, and the unit was started and tripped several

times. The data identified a torsional resonant frequency at approximately 23 Hz. This resonant frequency was found to be extremely close to the blade-passing frequency of the fan system (24 Hz.), indicating the possibility that the torsional resonance excited by the blade-passing perturbations was a contributing cause of the shaft failure. In an effort to solve the problem, the system was simulated on the eigenvector-eigenvalue computer program. The gear stiffness was computed as outlined, and the calculated torsional resonant frequency was in good agreement with the field data. To demonstrate the system sensitivity to gear stiffness, a parametric analysis was made by varying the gear stiffness. This data is presented in Table **III.** It should be noted that the gear stiffness is important in the design stage where coincidence of resonant frequency and excitation is to be avoided.

RECIPROCATING ENGINES

Theory

A comprehensive torsional analysis of a reciprocating engine driven system must properly simulate the **mass-**elastic properties of the crankshaft and the force produced by the power cylinders. Simulation of the engine crankshaft stiffnesses and masses, although tedious, can be accomplished with careful calculations [1,2]. The forced vibration response, however, is less straightforward since it must include cylinder phasing (firing order), gas torques (harmonic content), and reciprocating mass inertia in combination with the dynamic response (mode shape). Gas torque curves are usually available for new engines in harmonic content form which can be directly used in the analysis[3].

The harmonic torque amplitudes for each cylinder are applied to the rotating mass at each crank thrown by expressing the torque function as a series of sine and cosine terms including phase angles. In this form, the forcing functions are easily used in the eigenvector matrix method which intrinsically combines the forcing function with the dynamic response to produce the vibration amplitudes and stresses.

By utilizing the **matrix** generated from the equation of motion, in combination with the applied forces, the torsional deflections can be calculated. **A** damping term must be included to calculate a dynamic magnification factor (Q) which provides a relationship between vibrational amplitude and frequency ration (ω/ω_n). The

vibrational amplitude can be simply expressed by the following expression:

$$[\theta] = Q\,\overline{F}\,[K]^{-1}$$

This technique has been used to determine reliability of reciprocating engine driven systems in the design stage as well as in operating systems. The following operating unit, plagued by shaft failures, was analyzed by this technique to determine the cause of failures.

Field Case

A plant air compressor system experienced several shaft failures because of operating near system torsional natural frequencies. The system consists of a **four-cyliner** air compressor driven by a **16-cylinder** reciprocating engine through a gear box. The mass-elastic properties of the system were calculated, and the torsional system was simulated with the **eigenvector-eigen-**value method. The calculated torsional natural frequencies of interest are as follows:

$$f_1 = 2688 \text{ cpm}$$

$$f_2 = 4212 \text{ cpm}$$

$$f_3 = 6180 \text{ cpm}$$

An initial analysis was made to study the effect of the engine torques on the system. The forced vibration analysis included both the engine and compressor loading torques which were obtained from the equipment manufacturers.

The maximum stresses for this system occurred in the pinion shaft and are plotted in Figure 2 for the engine harmonics and the second, third and fourth compressor harmonics in the speed range of the unit. The fourth compressor harmonic excites the first torsional natural frequency at an engine speed of 1206 cpm. The tird harmonic excites the first torsional natural frequency at an engine speed of 1608 cpm. The second compressor harmonic is on the flank of the reasonance curve, and these stresses increase from 1700 to 3000 psi **peak-to-**peak. The combined maximum stresses were caused by

compressor loading torques at 1608 cpm and were slightly in excess of 14800 psi peak-to-peak which exceeds the U. S. MIL STD* criteria for torsional stress levels. The maximum stresses produced by the engine were 10,000 psi peak-to-peak when the fourth order reacted with the second torsional natural frequency.

The Campbell diagram or interference diagram shown in Figure 3 indicates which engine and compressor harmonics excite the torsional natural frequencies through the speed range. The second engine order and the third and fourth compressor harmonics excite the first torsional natural frequency (Figure 2). These produce the major contributing stresses which act on the system. The second torsional natural frequency (4212 cpm), however, can be excited by the fifth, sixth, and seventh compressor harmonics as well as the third and fourth engine orders. The effect of the higher harmonics exciting the second torsional natural frequency serves to increase the stresses of the pinion shaft also. The third torsional natural frequency (6180 cpm) can be excited by the seventh through eleventh compressor harmonics as well as the fourth through sixth engine orders. These higher harmonic torques will also contribute additional stress; however, these harmonics produce a small percentage of energy compared to the first few harmonics and consequently do not cause a significant increase in stress level.

DISCUSSION AND CONCLUSION

Proper design analysis should be implemented on rotary equipment to insure increased reliability and minimal downtime due to excessive maintenance or failures. One area of major concern is the location of torsional resonant frequencies relative to the excitation forces in the system. The eigenvector-eigenvalue method provides the following information:

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*The allowable torsional stress for this system based upon U.S. Navy MIL STD 167 is 4000 psi zero peak or 8000 psi peak-to-peak

Calculation of all possible resonant frequencies directly.

Exact simulation of multiple-branched systems is possible along with branch-on-branch capabilities.

Forcing functions can be applied to any mass location while maintaining proper phase relationships.

Calculation of torsional stresses produced by unbalanced torques for complex systems.

Since all calculations are dependent on the initial values, it is imperative that rotary mass, stiffness, and damping be properly calculated and applied. Gear stiffness has classically plagued the accuracy of torsional calculations. However, the eigenvector-eigenvalue technique along with proper simulation of stiffness, mass and damping has been used to advantage in solving problems in operating systems.

NOMENCLATURE

A, B, C, D, E. F	=	Complex functions of mass and stiffness.
$\overline{F}$	=	Force vector (torques in/lbs).
G	=	Torsional Modulus (lb/in^2).
h	=	Height of triangle in Figure I (in).
h_p	=	Height from chord to pitch circle in Figure I (in).
$[J]$	=	Diagonalized mass matrix.
$[K]$	=	Stiffness matrix.
K	=	Linear stiffness of gear tooth.
K_F	=	Rotational stiffness related to a gear wheel.
K_{total}	=	Total linear stiffness.
K_1, K_2	=	Linear stiffness of driver and driver gear.
K_1', K_2'	=	Linear stiffness after the corrections factors are applied.
L	=	Face width of gear tooth (in).
Q	=	Dynamic magnification factor.
R	=	Correction factor for secondary effects.
r	=	Gear wheel radius (in).
T	=	Length of chord at root of tooth (in).
Y	=	Modulus of elasticity (lb/in^2).
$[\theta]$	=	Angular displacement matrix.
$[\ddot{\theta}]$	=	Angular acceleration matrix.
$[\lambda]$	=	Diagonalized eigenvalue matrix.
ω	=	Operating frequency, radians per second.
ω_n	=	Torsional natural frequency, radians per second.
$[\omega^2]$	=	Diagonalized eigenvalue matrix.

REFERENCES

1. Ker Wilson, W., Practical Solution of Torsional Vibration Problems, Vol. I, (1956), Vol. II, (1963) (London: Chapman and Hall).

2. Nestorides, E. J., (Editor) A Handbook on Torsional Vibration (London: Cambridge University Press), 1958.

3. Porter, Frederic P., "Harmonic Coefficients of Engine Torque Curves", Journal of Applied Mechanics, March 1943, pp A-33.

TABLE I

TORSIONAL EQUATIONS OF MOTION

$$J_1 \ddot{\theta}_1 = K_1 (\theta_2 - \theta_1)$$

$$J_2 \ddot{\theta}_2 = K_1 (\theta_1 - \theta_2) + K_2 (\theta_3 - \theta_2)$$

$$J_3 \ddot{\theta}_3 = K_2 (\theta_2 - \theta_3) + K_3 (\theta_4 - \theta_3)$$

$$J_4' \ddot{\theta}_4 = K_3 (\theta_3 - \theta_4) + K_4' (\theta_5 - \theta_6)$$

$$J_5' \ddot{\theta}_5 = K_4' (\theta_4 - \theta_5)$$

$$J_4' = J_{4A} + n^2 J_{4B}$$

$$J_5' = n^2 J_5$$

$$K_4' = n^2 K_4$$

TABLE II

TORSIONAL ANALYSIS EIGENVALUE MATRIX EQUATION

$$
\begin{bmatrix}
J_1 & 0 & 0 & 0 & 0 \\
0 & J_2 & 0 & 0 & 0 \\
0 & 0 & J_3 & 0 & 0 \\
0 & 0 & 0 & J_4 & 0 \\
0 & 0 & 0 & 0 & J_5'
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta}_1 \\
\ddot{\theta}_2 \\
\ddot{\theta}_3 \\
\ddot{\theta}_4 \\
\ddot{\theta}_5
\end{bmatrix}
=
\begin{bmatrix}
K_1 & -K_1 & 0 & 0 & 0 \\
-K_1 & K_1+K_2 & -K_2 & 0 & 0 \\
0 & -K_2 & K_2+K_3 & -K_3 & 0 \\
0 & 0 & -K_3 & K_3+K_4 & -K_4' \\
0 & 0 & 0 & -K_4' & K_4'
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5
\end{bmatrix}
$$

TABLE III

RESONANT FREQUENCY PARAMETRIC ANALYSIS

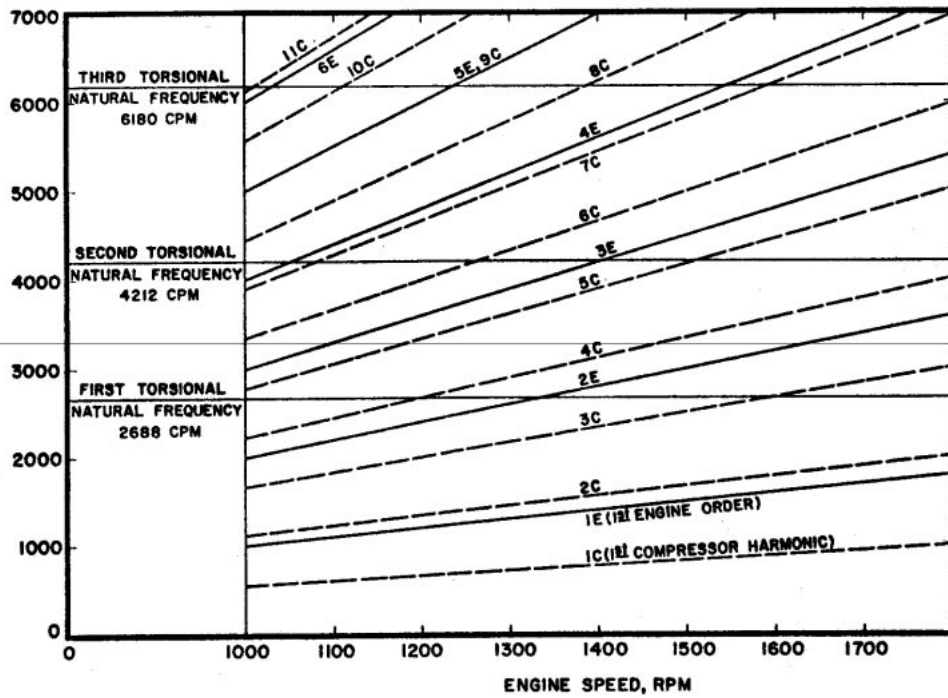| GEAR STIFFNESS (in-lb/rad)	TORSIONAL RESONANT FREQUENCY Hz
0.1×10^{6}	4.94
1.0×10^{6}	13.85
3.8×10^{6}	20.8
6.24×10^{6}	22.99
10.0×10^{6}	24.62
100.0×10^{6}	27.77
$1,000.0 \times 10^{6}$	28.13
$10,000.0 \times 10^{6}$	28.16
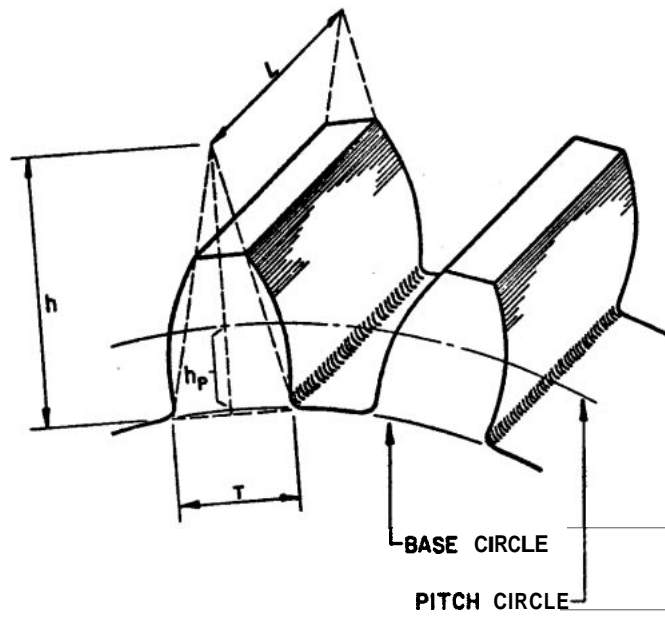
GEOMETRIC NOMENCLATURE

FIGURE I
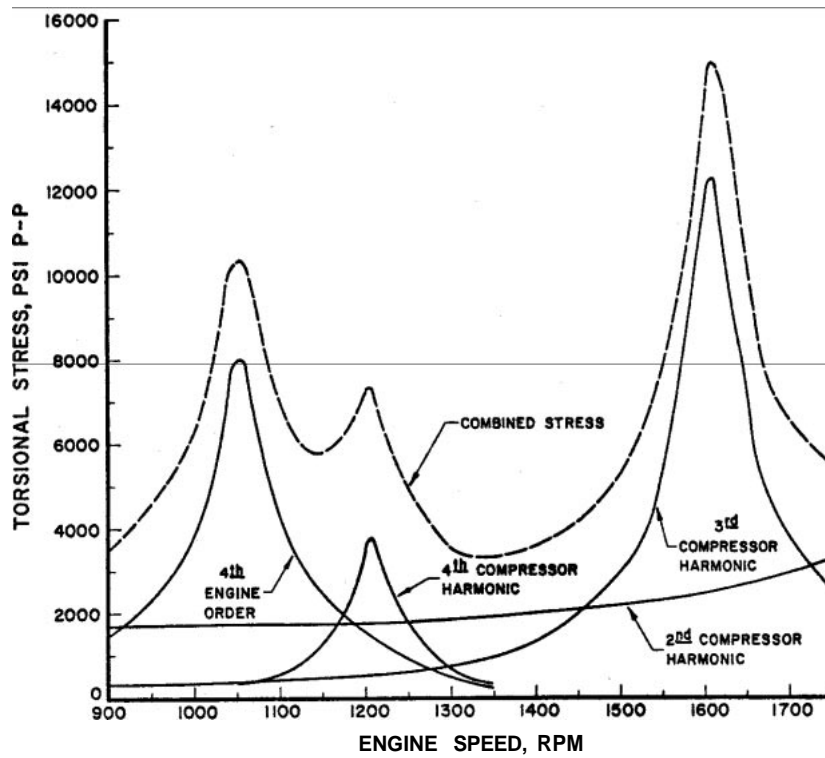
CALCULATED PINION SHAFT STRESSES.

FIGURE 2

INTERFERENCE DIAGRAM OF ENGINE ORDERS (E) AND
COMPRESSOR HARMONICS (C) WITH TORSIONAL RESONANCES.

FIGURE 3